The hypertree poset

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Motivation : $P\Sigma_n$

- F_n generated by $(x_i)_{i=1}^n$
- $P\Sigma_n$, *pure symmetric automorphism group*
  - group of automorphisms of F_n which send each x_i to a conjugate of itself,
  - group of motion of a collection of n unknotted, unlinked circles in 3-space.
- Use of the hypertree poset for the computation of the l^2-Betti numbers of $P\Sigma_n$ by C. Jensen, J. McCammond and J. Meier.
- It seems that their cohomology groups are not Koszul (A. Conner and P. Goetz).

Sommaire

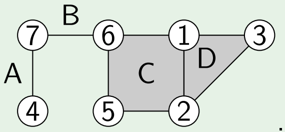
Hypergraphs and hypertrees

Definition ([Ber89])

A hypergraph (on a set V) is an ordered pair (V, E) where:

- *V is a finite set (vertices)*
- *E is a collection of subsets of cardinality at least two of elements of V (edges).*

example of a hypergraph on $[1;7]$



Walk on a hypergraph

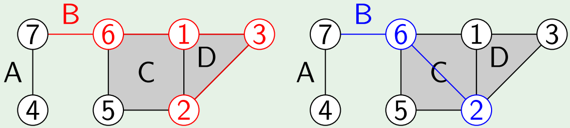
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks



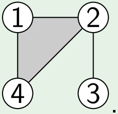
Hypertrees

Definition

A hypertree is a nonempty hypergraph H such that, given any distinct vertices v and w in H,

- *there exists a walk from v to w in H with distinct edges e_i, (H is connected),*
- *and this walk is unique, (H has no cycles).*

Example of a hypertree



The hypertree poset

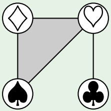
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

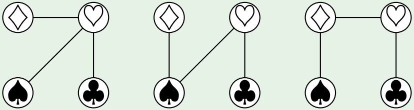
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



$$\preceq$$



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- Graded poset by the number of edges [McCullough and Miller],
- There is a unique minimum $\hat{0}$,
- $\mathrm{HT(I)}$ = hypertree poset on I,
- $\mathrm{HT_n}$ = hypertree poset on $[1, n]$.

Goal:

- New computation of the homology dimension
- Computation of the action of the symmetric group on the homology

What are species?

Definition

A species F *is a functor from the category of finite sets and bijections to itself. To a finite set* I*, the species* F *associates a finite set* $\mathrm{F}(I)$ *independent from the nature of* I*.*

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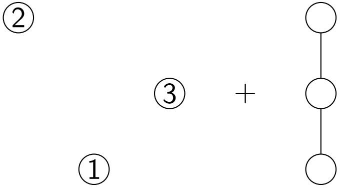

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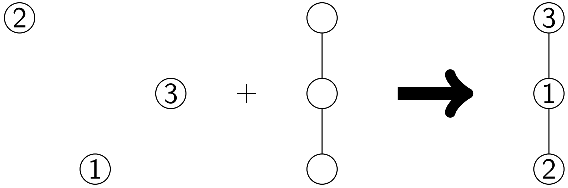

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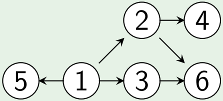
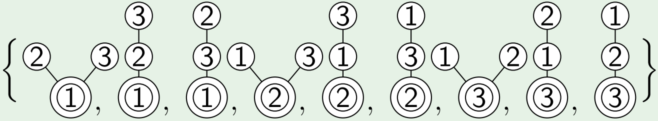
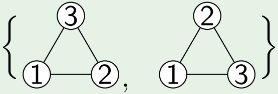
Counterexamples

The following sets are *not* obtained using species:

- $\{(1,\mathbf{3},2),(2,1,\mathbf{3}),(2,\mathbf{3},1)(3,1,\mathbf{2})\}$(set of permutations of $\{1,2,3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1,2,3,4,5,6\}$)



Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists on $\{1,2,3\}$)
- $\{\{1,2,3\}\}$ (species of non-empty sets Comm)
- $\{\{1\},\{2\},\{3\}\}$ (species of pointed sets Perm)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists on $\{\clubsuit,\heartsuit,\spadesuit\}$)
- $\{\{\heartsuit,\spadesuit,\clubsuit\}\}$ (Species of non-empty sets Comm)
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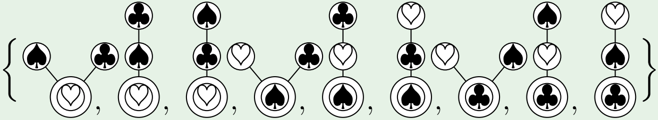
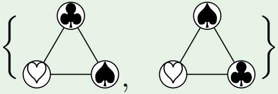
-  (Species of rooted trees PreLie)
-  (Species of cycles)

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Operations on species and generating series

Proposition

Let F and G be two species. The following operations can be defined on them:

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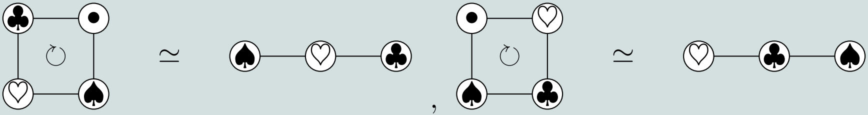
Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$



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- $(F \times G)(I) = \sum_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, (product)

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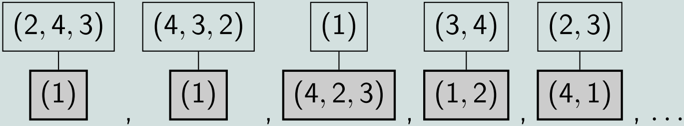
Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$



Definition

To a species F, we associate its generating series:

$$C_F(x) = \sum_{n\geq 0} \#F(\{1,\dots,n\})\frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\text{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\text{Comm}} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\text{Perm}} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\text{PreLie}} = \sum_{n\geq 0} n^{n-1}\frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{Cycles} = -\ln(1-x)$.

Homology of the poset

To each poset, we can associate a homology.

Definition

A strict k-chain of hypertrees on I is a k-tuple $(a_1, \ldots, a_k)$, where a_i are hypertrees on I different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.

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Let C_k be the vector space generated by strict $k+1$-chains. We define $C_{-1} = \mathbb{C}.e$. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 \prec \ldots \prec a_{k+1}) = \sum_{i=1}^{k}(-1)^i(a_1 \prec \ldots \prec \hat{a}_i \prec \ldots \prec a_k),$$

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$(a_1 \prec \ldots \prec a_{k+1}) \in C_k$.
The homology is defined by:

$$\tilde{H}_j = ker\partial_j / im\partial_{j+1}.$$

Theorem ([MM04])

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Corollary

The character for the action of the symmetric group on $\tilde{H}_{n-3}$ *is given in terms of characters for the action of the symmetric group on* C_k *by:*

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I.$$

Counting strict chains using large chains

Let I be a finite set of cardinality n.

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Let $M_{k,s}$ be the set of words on $\{0,1\}$ of lenght k, with s letters "1". The species $\mathcal{M}_{k,s}$ is defined by:

$$\begin{cases} \emptyset & \mapsto & M_{k,s}, \\ V \neq \emptyset & \mapsto & \emptyset. \end{cases}$$

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Proposition

The species $\mathcal{H}_k$ of large k-chains and $\mathcal{HS}_i$ of strict i-chains are related by:

$$\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$

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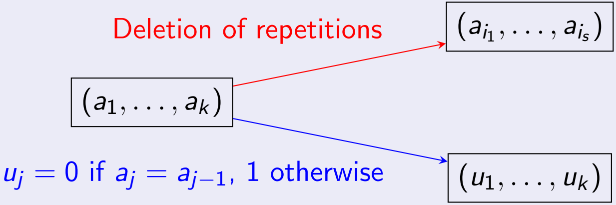
Proof.



with $a_0 = \hat{0}$. □

The previous proposition gives, for all integer $k > 0$:

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χ_k is a polynomial $P(k)$ in k which gives, once evaluated in -1, the character:

Corollary

$$\chi_{\tilde{H}_{n-3}} = (-1)^n P(-1) =: (-1)^n \chi_{-1}$$

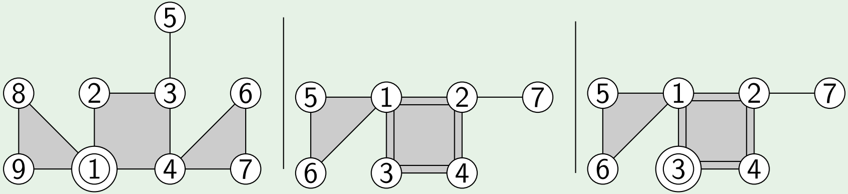
The hypertrees will now be on $\{1, \ldots, n\}$.

Definition

Let H be a hypertree on I. H is:

- *rooted in a vertex s if the vertex s of H is distinguished,*
- *edge-pointed in an edge a if the edge a of H is distinguished,*
- *rooted edge-pointed in a vertex s in an edge a if the edge a of H and a vertex s of a are distinguished.*

Example of pointed hypertrees



Proposition: Dissymmetry principle

The species of hypertrees and of rooted hypertrees are related by:

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}.$$

We write:

- $\mathcal{H}_k$, the species of large k-chains of hypertrees,
- $\mathcal{H}_k^{pa}$, the species of large k-chains of hypertrees whose minimum is rooted edge-pointed,
- $\mathcal{H}_k^{p}$, the species of large k-chains of hypertrees whose minimum is rooted,
- $\mathcal{H}_k^{a}$, the species of large k-chains of hypertrees whose minimum is edge-pointed.

Corollary ([Oge13])

The species of large k-chains of hypertrees are related by:

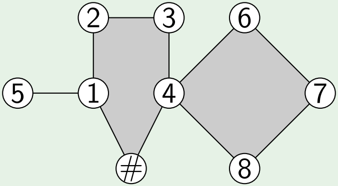
$$\mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^{p} + \mathcal{H}_k^{a}.$$

Last but not least type of hypertrees

Definition

A hollow hypertree on n vertices ($n \geq 2$) *is a hypertree on the set* $\{\#, 1, \ldots, n\}$*, such that the vertex labelled by* $\#$*, called the gap, belongs to one and only one edge.*

Example of a hollow hypertree



We denote by $\mathcal{H}_k^{cm}$, the species of large k-chains of hypertrees whose minimum is a hollow hypertree with only one edge and by $\mathcal{H}_k^{c}$, the species of large k-chains of hypertrees whose minimum is a hollow hypertree.

Relations between species of hypertrees

Theorem

The species $\mathcal{H}_k$, $\mathcal{H}_k^p$, $\mathcal{H}_k^c$ and $\mathcal{H}_k^{cm}$ satisfy:

$$\mathcal{H}_k^p = X \times \mathcal{H}_k' \tag{1}$$

$$\mathcal{H}_k^p = X \times \mathrm{Comm} \circ \mathcal{H}_k^c + X, \tag{2}$$

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \tag{3}$$

$$\mathcal{H}_k^{cm} = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c. \tag{4}$$

$$\mathcal{H}_k^a = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p. \tag{5}$$

$$\mathcal{H}_k^{pa} = \left(\mathcal{H}_{k-1}^p - x\right) \circ \mathcal{H}_k^p. \tag{6}$$

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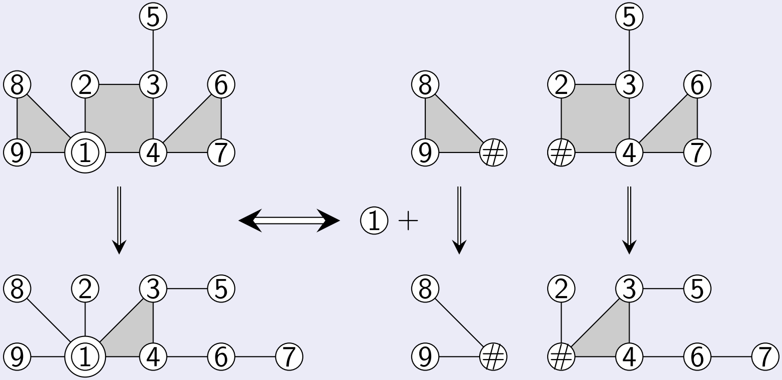
Proof.

1. Rooting a species F is the same as multiply the singleton species X by the derivative of F,

Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

$$\mathcal{H}_k^p = X \times \text{Comm} \circ \mathcal{H}_k^c + X,$$

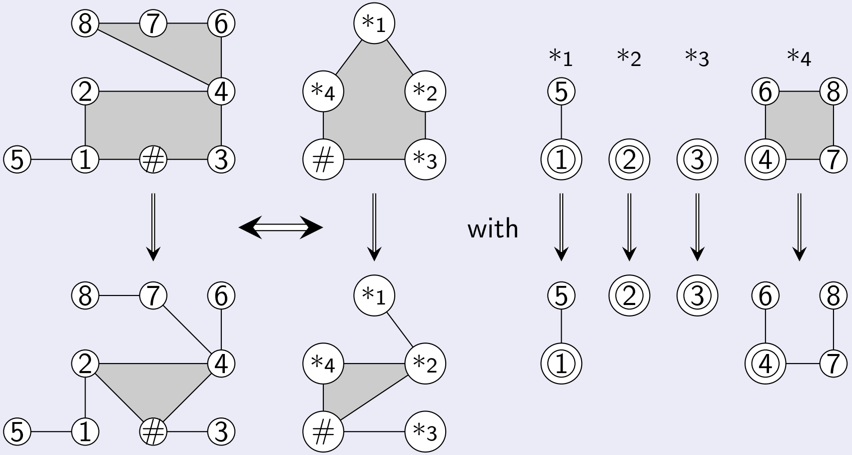


and End!

3. Hollow case:

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \tag{7}$$

$$\mathcal{H}_k^{cm} = \text{Comm} \circ \mathcal{H}_{k-1}^c. \tag{8}$$



Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k$, $\mathcal{H}_k^p$, $\mathcal{H}_k^c$ and $\mathcal{H}_k^{cm}$ satisfy:

$$\mathcal{C}_k^p = x \cdot \exp\left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1\right), \tag{9}$$

$$\mathcal{C}_k^a = (\mathcal{C}_{k-1} - x)(\mathcal{C}_k^p), \tag{10}$$

$$\mathcal{C}_k^{pa} = (\mathcal{C}_{k-1}^p - x)(\mathcal{C}_k^p), \tag{11}$$

$$x \cdot \mathcal{C}_k' = \mathcal{C}_k^p, \tag{12}$$

$$\mathcal{C}_k + \mathcal{C}_k^{pa} = \mathcal{C}_k^p + \mathcal{C}_k^a. \tag{13}$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^p = xe^x.$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^p = xe^x.$$

This implies with the previous theorem:

Theorem ([MM04])

The dimension of the only homology group of the hypertree poset is $(n-1)^{n-2}$.

This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.

1. This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.
   The operad (a species with more properties on substitution) whose vector space are PreLie(n) is PreLie.
2. This operad is anticyclic (cf. [Cha05]): There is an action of the symmetric group $\mathfrak{S}_n$ on PreLie($n-1$) which extends the one of $\mathfrak{S}_{n-1}$.
3. Moreover, there is an action of $\mathfrak{S}_n$ on the homology of the poset of hypertrees on n vertices.

Question

Is the action of $\mathfrak{S}_n$ on PreLie(n-1) the same as the action on the homology of the poset of hypertrees on n vertices?

Definition

The cycle index series of a species F of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- *with F^σ, is the set of F-structures fixed under the action of σ,*
- *and σ_i, the number of cycles of length i in the decomposition of σ into disjoint cycles.*

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1-p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\text{Comm}} = \exp(\sum_{k \geq 1} \frac{p_k}{k}) - 1$.

Operations

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$\begin{aligned} Z_{F+G} &= Z_F + Z_G, & Z_{F\times G} &= Z_F \times Z_G, \\ Z_{F\circ G} &= Z_F \circ Z_G, & Z_{F'} &= \frac{\partial Z_F}{\partial p_1}. \end{aligned}$$

Definition

The suspension Σ_t of a cycle index series $f(p_1, p_2, p_3, \dots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \dots).$$

Using relations on species established previously, we obtain:

Proposition

The series Z_k, Z_k^p, Z_k^a and Z_k^{pa} satisfy the following relations:

$$Z_k + Z_k^{pa} = Z_k^p + Z_k^a, \tag{14}$$

$$Z_k^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z_{k-1}^p \circ Z_k^p - Z_k^p}{Z_k^p} \right), \tag{15}$$

$$Z_k^a + Z_k^p = Z_{k-1} \circ Z_k^p, \tag{16}$$

$$Z_k^{pa} + Z_k^p = Z_{k-1}^p \circ Z_k^p, \tag{17}$$

and

$$p_1 \frac{\partial Z_k}{\partial p_1} = Z_k^p. \tag{18}$$

Theorem ([Oge13], conjecture of [Cha07])

The cycle index series Z_{-1}, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series M associated with the anticyclic structure of PreLie *by:*

$$Z_{-1} = p_1 - \Sigma M = \mathrm{Comm} \circ \Sigma \, \mathrm{PreLie} + p_1 \left(\Sigma \, \mathrm{PreLie} + 1 \right). \qquad (19)$$

The cycle index series Z^p_{-1} is given by:

$$Z^p_{-1} = p_1 \left(\Sigma \, \mathrm{PreLie} + 1 \right). \qquad (20)$$

Theorem ([Oge13], conjecture of [Cha07])

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Proof.

Sketch of the proof

1. Computation of $Z_0 = \mathrm{Comm}$ and $Z^p_0 = \mathrm{Perm} = p_1 + p_1 \times \mathrm{Comm}$
2. Replaced in the formula giving Z^p_0 in terms of itself and Z^p_{-1}

$$Z^p_0 = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z^p_{-1} \circ Z^p_0 - Z^p_0}{Z^p_0} \right),$$

Second part of the proof.

3. As $\Sigma\,\mathrm{PreLie} \circ \mathrm{Perm} = \mathrm{Perm} \circ \Sigma\,\mathrm{PreLie} = p_1$, according to [Cha07], we get:

$$Z^{p}_{-1} = p_1 \left(\Sigma\,\mathrm{PreLie} + 1 \right).$$

4. The dissymetry principle associated with the expressions gives:

$$\mathrm{Comm} + Z^{p}_{-1} \circ \mathrm{Perm} - \mathrm{Perm} = \mathrm{Perm} + Z_{-1} \circ \mathrm{Perm} - \mathrm{Perm}.$$

5. Thanks to equation [Cha05, equation 50], we conclude:

$$\Sigma M - 1 = -p_1\left(-1 + \Sigma\,\mathrm{PreLie} + \frac{1}{\Sigma\,\mathrm{PreLie}}\right).$$

□

Thank you for your attention !

[Oge13] Bérénice Oger *Action of the symmetric groups on the homology of the hypertree posets*. Journal of Algebraic Combinatorics, february 2013.

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Eccentricity

Definition

The eccentricity of a vertex or an edge is the maximal number of vertices on a walk without repetition to another vertex.
The center of a hypertree is the vertex or the edge with minimal eccentricity.

Example of eccentricity